THE LOGIC OF OPINION

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Abstract: Unlike truth values that, accordingly to the principles of logic, are only two, there can be distinguished three values of opinion for a proposition: *acceptance*, *rejection* and *indifference*. Still, the attempt to build a three-valued logic of opinion comes to the paradox that there are tautologies that are not accepted. We can avoid the paradox if we use a complex representation of the opinion values, through analogy with complex numbers. The logic of opinion is a kind of *complex logic*.

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Through "opinion" we will understand the attribution of a truth value to a proposition. For instance, someone has the opinion that *Earth is flat* if and only if he considers that the proposition "Earth is flat" is true. The value associated to a proposition can be different of the real truth value of that proposition; consequently, there are wrong opinions.

The opinions are freely formed, without constraints, because they take place in the present time. We'll take into consideration a *rational* subject, (*S*), subdued only to the principles of logic.[1] According to the excluded middle principle there are only two truth values, namely, *true* and *false* and, from the principle of contradiction, results that a proposition cannot be both true and false. Therefore, there are only three rational choices concerning attribution of the truth values to a proposition, *p*: a) *p* is considered true; b) *p* is considered false; c) the subject refrains to attribute a truth value for *p*. The last alternative doesn't mean that *p* has no truth value but just that the subject is undecided about it.[2]

If a proposition is considered true, we'll say that it is *accepted*; if it is considered false, then the proposition is *rejected*; finally, in the case (c), the proposition is called *indifferent*. Opinions divide propositions into three disjoint classes, accepted, rejected and indifferent, while, following the criterion of truth values, there are only two categories of propositions:

[1] Garson J. W., *2006*, p. 54.
[2] Beal J. C., Fraasen B. C. v., *2003*, p. 107.

true and false. Similarly to the case of truth values, we can distinguish three *opinion values* of propositions, namely, *acceptance,* (A), *rejection,* (R), and *indifference,* (I). Consequently, while the logic of truth values is *bivalent*, the logic of opinion[3] has to be at least a *three-valued* logic.

In order to define the negation in the logic of opinion, let's start from the observation that a rational subject can't give the same truth value both to a proposition and its negation. Therefore, if a proposition is accepted, the negation of that proposition has to be rejected and reciprocally. In the same time, if a proposition is indifferent then its negation is indifferent too, obtaining the following table:

p	~p
A	R
R	A
I	I

Concerning the disjunction, the acceptance is an absorbing element, while the rejection is the neutral element:

p v q	A	R	I
A	A	A	A
R	A	R	I
I	A	I	I

If a proposition is accepted, it means that it is considered true, when the disjunction must also be considered true, while, if a proposition is rejected, the subject attributes to it the value false, namely, such a proposition doesn't change the truth value of disjunction and, also, the opinion value of disjunction. For instance, if the proposition "Earth is flat" is accepted by a rational subject, and the proposition "Earth spins around its axis" is rejected, then their disjunction is rationally accepted.

Using previous definitions of negation and disjunction, we may introduce other *opinion's functions*. For instance, the definition tables for *conjunction* and *implication* are built keeping into account the relations: "(p & q) = ~(~p v ~q)" and "(p ⊃ q) = (~p v q)":

[3] Formigari L., *2009* , p. 7.

p&q	A	R	I
A	A	R	I
R	R	R	R
I	I	R	I

p⊃q	A	R	I
A	A	R	I
R	A	A	A
I	A	I	I

For conjunction, the acceptance is the neutral element and the rejection plays the role of absorbing element. Implication must be accepted if the antecedent is rejected or if its consequent is accepted. We notice that, although "A ⊃ A" and "R ⊃ R" have to be accepted, "I ⊃ I" remains indifferent because, in such a case, there is no distinction among the truth values of propositions and, consequently, there can't be excluded the version that the antecedent is true and the consequent is false. For that reason, there is no justification to consider the implication between two indifferent propositions as acceptable. In fact, in conformity with the table of negation and the mentioned definition of implication, we reach the result: (I ⊃ I) = (~I v I) = (I v I) = I.

A formula is *valid* in the logic of opinion if and only if, for any interpretations of its variables, their value is acceptance. For a rational subject, the logical laws should be valid in the logic of opinion because their acceptance is imposed by the principles of logic; for instance, any propositions with the structure "p v ~p" should be accepted, no matter the opinion about propositions *p*.

If we decide upon the formula E = "p v ~p" using the previous definitions of the opinion functions, we reach the result that it is not valid in the opinion logic:

p	~p	pv~p
A	R	A
R	A	A
I	I	I

We notice that, although the formula *E* is a logical law, it is indifferent in the logic of opinion if *p* receives the interpretation *indifferent*, despite the fact that all its interpretations are true propositions so, rationally, they should be accepted without exceptions. The reason of the opinion logic's failure is the way that the negation was defined. This paradoxical situation (we'll call it *the paradox of opinion logic*) follows from the fact that the negation of indifference is the indifference. Despite these, we cannot admit that the negation of indifference to be acceptance or rejection. If we followed the first variant then, the unrealizable expression “p & ~p” wouldn’t be rejected by the logic of opinion and, in the second situation, *E* would remain invalid.

We have to admit that the negation of indifference can be none of the three opinion values and we must define another value for it, with the consequence that the opinion logic cannot be a three-valued logic.[4]

The negation problem in the logic of opinion can be solved taking into consideration that the indifference means indetermination between acceptance and rejection (namely, between the attribution of true or false to a proposition). By consequence, the indifference has to depend on a parameter with two values so that, for a value it shall become acceptance and for the other value, instead of indifference we will get the rejection. Of course, such a parameter is proper to every proposition and, using it, we can represent all opinion values. If, for the proposition *p*, the parameter is symbolized through *P*, the expression of the opinion value of the proposition *p* will be: <p> = AP + RP*, where <p> is “the opinion value of the proposition *p*” and P* is the negation of *P*.

We note with “1” and “0” the two possible values of the parameter *P*, and with “C”, the situation of indetermination. The operations of negation, sum and product for these values are defined by the next tables:

P	P*
1	0
0	1
C	C*

P+Q	1	0	C*
1	1	1	1
0	1	0	C*
C	1	C	1

PQ	1	0	C*
1	1	0	C*
0	0	0	0
C	C	0	0

Also, the relations: P** = P; (P* + Q*)* = PQ; (P*Q*)* = P + Q take place. The opinion values of *p* depend on the parameter *P* according to the following table:

P	P*	<p>
1	0	A
0	1	R
C	C*	I

We arrived to a representation of the opinion values which is similar with the complex numbers. The logic of opinion can be regarded like logic of complex values or, as a *complex*

[4] Lambert T. G., Minoro J., *2010*, p. 10.

logic, different of the many-valued logic. Inside of complex logic, the logical constants are defined as it follows:

Let $\langle p \rangle = AP + RP^*$ and $\langle q \rangle = AQ + RQ^*$ be the opinion values of the propositions *p* and *q*. In this case,

$\langle \sim p \rangle = AP^* + RP$

$\langle p \vee q \rangle = A(P + Q) + RP^*Q^*$

$\langle p \,\&\, q \rangle = \langle \sim(\sim p \vee \sim q) \rangle = A(P^* + Q^*)^* + R(PQ)^* = APQ + R(P^* + Q^*)$

$\langle p \supset q \rangle = \langle \sim p \vee q \rangle = A(P^* + Q) + RPQ^*$

Let's show that, using the complex opinion values, the above paradox of opinion logic is avoided. The calculus for the formula *E* runs in the following fashion:

$E = p \vee \sim p$

$\langle E \rangle = A(P + P^*) + RP^*P$

$\langle E \rangle = A$, for any interpretation of the symbol *p*,

E is valid inside of logic of opinion.

The logical law *E* is a valid formula in the opinion logic and it has to be accepted by a rational subject, eliminating the paradox. Similarly, a contradiction as G = "p & ~p" must be rejected for any value of the variable *p*:

$G = p \,\&\, \sim p$

$\langle p \,\&\, \sim p \rangle = APP^* + R(P^* + P) = R$

$\langle p \,\&\, \sim p \rangle = R.$

The parameter *P* can be used to give a measure for the *intensity* of an opinion. Let's suppose that *P* can have any values inside of closed interval [0, 1], where the expressions "1" and "0" are keeping their previous interpretation, namely, P = 1 means *acceptance* and P = 0 means *rejection*. Any intermediary value represents a degree of *indifference* closer or more distant from acceptance or rejection. In this situation, P = 1 represents the highest degree of acceptance, respectively, it represents the *certainty*, while P = 0 means the highest degree of rejection.[5] If, for instance, P = 1/4, and automatically P* = 3/4, (respectively, <p> = A(1/4) + R(3/4)) then a rational subject, *S*, rather rejects the proposition *p* than he accepts it. In this case, the degree of acceptance is 1/4 while the degree of rejection has the value 3/4. (We'll note the acceptance degree of proposition *p* through "Ap", namely, Ap = P. In our example, Ap = 1/4).

If P = P* = 1/2, the acceptance and rejection have the same strength and a rational subject cannot decide between them so that, the rational opinion value is indifference. The negation interchanges the acceptance and rejection weights. For the given example, <~p> = A(3/4) + R(1/4) – if the acceptance of a proposition is 1/4, then the acceptance of the negation is 3/4.

In order to calculate the opinion values parameter for a system composed from two propositions, we shall build the matrix of all consistent conjunctions between the two propositions and their negations:

C_3	C_2	C_1	C_0
p	p	~p	~p
q	~q	q	~q

The values of the opinion parameter are symbolized by C_i. Taking into account that the only constraints implied in the opinions generation are given by the principles of logic, we get the following equations system:

$$C_3 + C_2 = P$$
$$C_3 + C_1 = Q$$
$$C_0 + C_1 = P^*$$
$$C_0 + C_2 = Q^*.$$

In order to solve this system of equations, let's calculate its determinant:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 0$$

[5] Huber F., *2009*, p. 1.

Since the determinant of the equations system is zero, there is no determined solution, emphasizing one more time the opinions freedom. Even if the degree of the rational opinion for two propositions is given, there isn't a determined opinion for their conjunction; it can vary depending of the appreciation about the relation between the elements of conjunction.

We can express C_i relatively to any of them. For instance, let's note $C_3 = h$. In this case, we'll obtain:

$C_3 = h$

$C_2 = P - h$

$C_1 = Q - h$

$C_0 = (1 + h) - (P + Q)$

Using the property $C_i \in [0, 1]$, we get the following constraints for *h*:

$h \leq P; h \leq Q$

$h \geq 0; h \geq (P + Q) - 1$

We notice that *h* takes any value from *0* to the smallest between *P* and *Q*. Let's study the limit cases of the parameter *h*:

1) If $h = 0$, then $C_3 = 0$; *p* and *q* are considered *contrary*;

2) If $h = P$, then $C_2 = 0$; *q* is seen as a *consequence* of *p*;

3) If $h = Q$, then $C_1 = 0$; this time, the subject believes that *p* is a *consequence* of *q*;

4) For $h = (P + Q) - 1$, $C_0 = 0$; the propositions *p* and *q* are considered *subcontrary*;

5) If $h = 0$ and $P + Q = 1$, then $C_3 = C_0 = 0$; the subject's opinion is that *p* and *q* are *contradictory*;

6) Finally, if $h = P = Q$, then $C_2 = C_1 = 0$; the propositions *p* and *q* are considered *equivalent*.

If *S* considers that *p* and *q* are independent propositions, then the parameters C_i take the same value:

$h = P - h = Q - h = (1 + h) - (P + Q) = 1/4$, it follows that:

$P = Q = 1/2$ (the opinion that two propositions are independent implies the highest degree of indifference relatively to them).

Instead, if two propositions are indifferent, they aren't always independent:

If $P = Q = 1/2$ then it follows:

$C_3 = h$;

$C_2 = 1/2 – h$;

$C_1 = 1/2 – h$;

$C_0 = (1 + h) – 1 = h$, where $h \in [0, 1/2]$. The two indifferent propositions are also independent only for h = 1/4. If h = 0, they are contradictory and, if h = 1/2, *p* and *q* are equivalent. For instance, some subjects may consider that the conjunction of two propositions is false (h = 0) but they are not sure about the truth value of each proposition. Still, they have to admit that the propositions can't have the same truth value (they have to be contradictory).

If *p* and *q* are independent, then A(p & q) = ApAq, namely, the acceptance degree for the conjunction of two independent propositions is equal with the product of the acceptance degrees of the two propositions.

In order to calculate the acceptance degree for a formula, this formula is brought to its normal form. For example, let's find the acceptance degree for the formula K = (p v q) ⊃ (p & q). The normal form is:

K = (~p & ~q) v (p & q). It follows that

$AK = C_0 + C_3 = 1 + h – (P + Q) + h = (1 + 2h) – (P + Q)$.

For instance, if Ap = 1/3 and Aq = 1/4, then AK = (1 + 2h) – (1/3 + 1/4) = 5/12 + 2h, where h ≤ 1/4. If *S* has the opinion that *p* and *q* are contrary (h = 0), then AK = 5/12 and, if *q* is admitted as a consequence of *p* (h = 1/4), then AK = 11/12. The certainty is achieved when P + Q = 2h, respectively, when (P + Q) – h = h. In this case, A(p v q) = A(p & q). On the other hand, (P – h) + (Q – h) = 0, so that, P = Q = h. (*p* and *q* are equivalent).

Let us calculate now the acceptance degree of implication.[6] Since the normal form of the formula "p ⊃ q" is "(p & q) v (~p & q) v (~p & ~p)", the acceptance degree is:

$A(p \supset q) = C_3 + C_1 + C_0 = h + Q – h + 1 + h – P – Q$

$A(p \supset q) = 1 + h – P$

[6] Josang A., *2008*, p. 5.

$A(p \supset q) = h + P^*$

$A(p \supset q) = A(p \& q) + A(\sim p)$

The subject has the certainty that "$p \supset q$" when $A(p \& q) = Ap$, namely, if he has the same trust in antecedent and in the conjunction of antecedent and consequent. The rejection of an implication is equivalent with the rejection of the consequent and the acceptance of antecedent.

If an implication is accepted then the acceptance degree of the antecedent is lower than the acceptance degree of consequent:

If $A(p \supset q) = 1$ then $1 + h - P = 1$. It results:

$P = h$

Since $h \leq Q$, we reach the relation $P \leq Q$.

Therefore, if an inference is accepted, then the conclusion has to be accepted in a higher degree than the premise. It follows that if the premise is accepted and the inference is, at its turn accepted, then the conclusion must be accepted too. On the other hand, if the conclusion of an accepted inference is rejected then the premise must be rationally rejected. This result can be obtained from the previous observation that $P \leq Q$, but it is acquired also, by calculus:

$1 + h - P = 1$ (the inference is certainly accepted)

$Q = 0 \mid\!- h = 0$, because $h \leq Q$.

$1 - P = 1$

$P = 0$.

Namely, the premise of an accepted inference has to be rationally rejected if its conclusion is also rejected.

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