Minimal Models, Characteristic Formulas and Public Announcements: Talking about Truths at Possible Worlds

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Abstract

The first half of this paper is expository, its main purpose being to introduce different notions of the model theory of modal logic (mainly the ones related to the notion of bisimulation) to philosophers. The second half will be concerned with introducing the framework of Public Announcement Logic (independently discovered by J. Plaza and J. Gerbrandy) and showing how we can talk at a certain possible world about truths at other worlds in this new logical setting. Although this can be done successfully in Hybrid Logic, the new logical apparatus will offer a solution that does not involve accepting ”nominals”(propositional atoms that denote possible worlds). Also, using this new way of talking about truths at possible worlds, I will present a formula that states at a possible world that the model is maximally contracted.
Keywords: *bisimulation*, *bisimulation contraction*, *Kripke model*, *minimal model*, *public announcement logic*, *characteristic formulas*

1 Kripke Models

I will assume knowledge of the language of propositional modal logic and the axioms of S5. The main tool for interpreting sets of modal formulas is the Kripke-model, defined as follows:

$\mathfrak{M} = (W, \{\longrightarrow_a | a \in A\}, \pi)$, where:

1. W is a finite, nonempty set of *possible worlds*, *states*, *points* or *nodes*. Although there is a vast literature on the metaphysics of possible worlds, its relevance for the metalogical properties of propositional modal logic is rather limited. However, this matter changes when it comes to first-order quantified modal logic. Different philosophical stances towards what possible worlds

are (are they real? are they mere stipulations?), the related problems of trans-world identity and the designation of proper names lead to different approaches to what a good interpretation for modal should be like. For an alternative to the Kripke-models, see David Lewis' counterpart theory.

2. $\{\longrightarrow_a | a \in A\}$ is a collection of binary relations (*accessibility relations* between the worlds of our model) indexed by the set of agents A: $\longrightarrow_{a\in A} \subseteq W \times W$, and:

3. The *valuation function* - the function that assigns sets of worlds (subsets of our domain, W) to our propositional atoms: $\pi : \mathsf{Atoms}(\mathfrak{L}) \to 2^W$. Note that 2^W is the powerset of W. This notation is not arbitrary: the cardinal of the powerset of any set P (the set of all subsets of P) is $2^{card(P)}$, where $card(P)$ is the cardinal of P.

Being that we defined our main tool for the construction of a semantics of modal logic, the Kripke-model, we can now define the satisfaction relation $\models$: $\models \subseteq (\mathfrak{M}, w) \times \mathsf{Form}(\mathfrak{L})$ as the smallest relation that respects the following conditions:

R0. All worlds satisfy $\top$ and none satisfies $\perp$.

R1. $\mathfrak{M}, w \models p$ iff $w \in \pi(p)$, for all $p \in W$ and for all $p \in \mathsf{Atoms}(\mathfrak{L})$.

R2. $\mathfrak{M}, w \models \neg\phi$ iff $\mathfrak{M}, w \not\models \phi$, for all $\phi \in \mathsf{Form}(\mathfrak{L})$.

R3. $\mathfrak{M}, w \models \phi \wedge \psi$ iff $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$, for all $\phi, \psi \in \mathsf{Form}(\mathfrak{L})$.

R4. $\mathfrak{M}, w \models \phi \vee \psi$ iff $\mathfrak{M}, w \models \phi$ or $\mathfrak{M}, w \models \psi$, $\forall \phi, \psi \in \mathsf{Form}(\mathfrak{L})$.

R5. $\mathfrak{M}, w \models \phi \to \psi$ iff $\mathfrak{M}, w \not\models \phi$ or $\mathfrak{M}, w \models \psi$, $\forall \phi, \psi \in \mathsf{Form}(\mathfrak{L})$.

R6. $\mathfrak{M}, w \models \Diamond\phi$ iff $\exists u : Rwu \wedge \mathfrak{M}, u \models \phi$, $\forall \phi \in \mathsf{Form}(\mathfrak{L})$.

R7. $\mathfrak{M}, w \models \Box\phi$ iff $\forall u : Rwu \Rightarrow \mathfrak{M}, u \models \phi$, $\forall \phi \in \mathsf{Form}(\mathfrak{L})$.

2 Bisimulation

For starters, I'll offer the definition of the simplest notion of equivalence between models (see [3], [17]):

Definition. Let $w \in \mathfrak{M}$ and $w' \in \mathfrak{M}'$ be two states of two models. We call the states w and w' *equivalent iff* they satisfy the same formulas:

$w \equiv w'$ iff $\{\phi \mid \mathfrak{M}, w \models \phi\} = \{\phi \mid \mathfrak{M}', w' \models \phi\}$, or:

$w \equiv w'$ iff $(\mathfrak{M}, w \models \phi \Leftrightarrow \mathfrak{M}', w' \models \phi)$

This relation is often called *the elementary equivalence* relation.

Definition. Two models $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent *iff* they satisfy the same formulas:

$\mathfrak{M} \equiv \mathfrak{M}'$ iff $\{\phi \mid \mathfrak{M} \models \phi\} = \{\phi \mid \mathfrak{M}' \models \phi\}$, or:

$\mathfrak{M} \equiv \mathfrak{M}'$ iff $(\mathfrak{M} \models \phi \Leftrightarrow \mathfrak{M}' \models \phi)$

2.1 The Bisimulation Relation Between Kripke-Models

Bibliographical indication: see [3], pp. 65-66.

Let's consider a new equivalence relation between two models: $\sim \subseteq W \times W'$. We'll say $\sim$ is a *bisimulation* if the following conditions hold:

(2.1) *Atomic harmony*: If $w \sim w'$, then w and w' satisfy the same propositional letters:

$w \sim w' \Rightarrow \forall p \in \mathsf{Atoms}(\mathfrak{L}) : \mathfrak{M}, w \models p \Leftrightarrow \mathfrak{M}', w' \models p$.

(2.2) *Zig*: If $w \sim w'$ and $w \longrightarrow v$, then $\exists v' \in W' : v \sim v'$ and $w' \longrightarrow' v'$.

(2.3) *Zag*: If $w \sim w'$ and $w' \longrightarrow' v'$, then $\exists v \in W : v \sim v'$ and $w \longrightarrow v$.

2.2 Linking Bisimulation and Elementary Equivalence: The Henessy-Milner Theorem

Proposition. If two models are bisimilar (there is a bisimulation linking them), they are equivalent (meaning that they verify the same formulas). We say that between two bisimilar models all information is preserved.

$\forall w \in W, \forall w' \in W' : (\mathfrak{M}, w) \sim (\mathfrak{M}', w') \Rightarrow (\mathfrak{M}, w) \equiv (\mathfrak{M}', w')$

Proof (see [17], p. 25, [3], p. 67). By induction on the length of any proposition ϕ:

1. *Basis*: Assuming that $w \sim w'$, the *atomic harmony* condition assures us that $w \models p \Leftrightarrow w' \models p, \forall p \in \mathsf{Atoms}(\mathfrak{L})$

2. *Induction Hypothesis*: Assuming that $w \sim w'$, we obtain that $\mathfrak{M}, w \models \phi \Leftrightarrow \mathfrak{M}', w' \models \phi$

3. *Induction Step*:

3. a) *The Case of Negation*. Let's assume that $\mathfrak{M}, w \vDash \neg\phi$. Then, $\mathfrak{M}, w \nvDash \phi$. From this, using the hypothesis of induction we arrive at: $\mathfrak{M}', w' \nvDash \phi$, so $\mathfrak{M}', w' \vDash \neg\phi$. As a conclusion, he have that if $w \sim w'$, then $\mathfrak{M}, w \models \neg\phi \Leftrightarrow \mathfrak{M}', w' \models \neg\phi$.

3. b) *The Case of Conjunction*. Let's assume that $\mathfrak{M}, w \models \phi \wedge \psi$. From the semantic rule **R3**, we have that $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$. By the hypothesis of induction we have that $\mathfrak{M}', w' \models \phi$ and $\mathfrak{M}', w' \models \psi$, and, by the semantic rule **R3**, we arrive at: $\mathfrak{M}', w' \models \phi \wedge \psi$.

3. c) *The Case of Necessity*. We'll prove that if $w \sim w'$, then, if $\mathfrak{M}, w \models \Box\phi$, then: $\mathfrak{M}', w' \models \Box\phi$.

Let's say that $\mathfrak{M}, w \models \Box\phi$. We arbitrarily choose a state u' of W', so that $w' \longrightarrow' u'$. Because $w \sim w'$, from *zag*, we have that $\exists u : w \longrightarrow u$ and $u \sim u'$. By the hypothesis induction, we have: $\mathfrak{M}, u \models \phi \Leftrightarrow \mathfrak{M}', u' \models \phi$. Because $\mathfrak{M}, w \models \Box\phi$, then $\mathfrak{M}, u \models \phi$, so $\mathfrak{M}', u' \models \phi$. Since u' was arbitrarily chosen, we have that for all u' in W' so that $w' \longrightarrow' u'$ it holds that $\mathfrak{M}', u' \models \phi$. By the definition of the necessity operator, we derive that $\mathfrak{M}', w' \models \Box\phi$.

In order to show that if $w \sim w'$, then, if $\mathfrak{M}', w' \models \Box\phi$, then $\mathfrak{M}, w \models \Box\phi$, we only have to reason analogously, this time using *zig*.

3. d) *The Case of the Possibility Operator*.

We'll show that for $w \sim w'$, if $\mathfrak{M}, w \models \Diamond\phi$, then $\mathfrak{M}', w' \models \Diamond\phi$.

If $\mathfrak{M}, w \models \Diamond\phi$, then $\exists u : w \longrightarrow u$, with $\mathfrak{M}, u \models \phi$. By *zig*, we have that $\exists u' : w' \longrightarrow' u'$ and $u \sim u'$. By the hypothesis induction, we know that $\mathfrak{M}', u' \models \phi$. From this we conclude that $\mathfrak{M}', w' \models \Diamond\phi$, by the semantic rule of the possibility operator.

In order to show that: if $w \sim w'$, then, if $\mathfrak{M}', w' \models \Diamond\phi$, then $\mathfrak{M}, w \models \Diamond\phi$, we can reason by analogy, using, this time, *zag*. $\Box$

The Henessy-Milner Theorem (see [3], pp. 69-70) If two image-finite models are equivalent, then they are bisimilar.

The property of *image-finiteness* means that each accessibility relation links a certain world with only a finite number of other worlds: $\{u \mid w \longrightarrow u\}$ is finite.

Proof. The idea of this proof is to show that the equivalence relation $\equiv$ between models (as defined at the beginning of the second section) is a bisimulation relation.

1. $\equiv$ has the property of *atomic harmony*. If $\mathfrak{M}, w \equiv \mathfrak{M}', w'$, then, by the definition of equivalence, the two of them satisfy the same propositions and atoms.

2. $\equiv$ has the property of *zig*. Let us assume that $\mathfrak{M}, w \equiv \mathfrak{M}', w'$ and that $w \longrightarrow u$. Let's presuppose, for the sake of contradiction, that $\nexists u' : (w' \longrightarrow' u' \wedge u \equiv u')$.

We'll consider the following set: $S' = \{u' \mid w' \longrightarrow' u'\}$, as the set of all states accessible by $\longrightarrow' \in \mathfrak{M}'$ from the state $w' \in \mathfrak{M}'$. We know that $w \longrightarrow u$, so $w \models \Diamond\top$ (because $\top$ is true at every world), which is equivalent to $w \models \neg\Box\bot$, so $w \not\models \Box\bot$. Because it holds that $w \equiv w'$, we have that $w \nvDash \Box\bot$, so $S' \neq \emptyset$ (the states in which it is true that $\Box\bot$ are the *dead-end* states - the ones that have no other state accessible).

Now, because $\mathfrak{M}'$ is *image-finite*, it follows that $S' \neq \emptyset$ is finite, so its elements can be enumerated, of course, since every finite set is an enumerable set: $S' = \{u'_1, ..., u'_n\}$. By assumption, we have that $\forall u'_i \in S', \exists \phi_i : \mathfrak{M}, u \vDash \phi_i$ and $\mathfrak{M}', u'_i \nvDash \phi_i$. Let's take each conjunct and see what we can infer from them:

From $\forall u'_i \in S', \exists \phi_i : \mathfrak{M}', u'_i \nvDash \phi_i$, we deduce that $\mathfrak{M}', u'_i \vDash \neg\phi_i$. Now, since in any state u'_i accessible from w' there is a formula ϕ_i so that $\mathfrak{M}', u'_i \vDash \neg\phi_i$, we have that $\forall u'_i \in S' \exists \phi_i : \mathfrak{M}', u'_i \vDash \bigvee_{i=1}^{n} \neg\phi_i$. Because $\bigvee_{i=1}^{n} \neg\phi_i$ is true in all worlds accessible to w', we deduce that $\mathfrak{M}', w' \vDash \Box(\bigvee \neg\phi_i) \Leftrightarrow \mathfrak{M}', w' \vDash \neg\Diamond\neg(\bigvee_{i=1}^{n} \neg\phi_i) \Leftrightarrow \mathfrak{M}', w' \vDash \neg\Diamond(\bigwedge_{i=1}^{n} \phi_i) \Leftrightarrow \mathfrak{M}', w' \nvDash \Diamond(\bigwedge_{i=1}^{n} \phi_i)$. (*)

From $\forall u'_i \in S', \exists \phi_i : \mathfrak{M}, u \vDash \phi_i$, we immediately obtain that $\mathfrak{M}, u \vDash \bigwedge_{i=1}^{n} \phi_i$, and that $\mathfrak{M}, w \vDash \Diamond(\bigwedge_{i=1}^{n} \phi)$, which contradicts our previous (*), and finally we see rejected our assumption that the *zig* property doesn't hold. So, $\equiv$ has the property *zig*.

3. $\equiv$ has the property of *zag*. We can proceed in analogy with the above (for *zig*). $\Box$

In conclusion, any two image-finite models are bisimilar if and only if they are equivalent.

2.3 Some Properties of Bisimulation:

a) Bisimulation is an *equivalence relation* between two models (an *equivalence relation* is a *reflexive*, *symmetrical*, and *transitive* relation).

Proof. To prove this, we simply have to show that bisimulation verifies the three properties of an equivalence relation, for an arbitrarily chosen model $\mathfrak{M} = (W, \longrightarrow, \pi)$.

1. *Reflexivity*: $a \sim a$. It is evident that a satisfies the same atoms as a (so we have *atomic harmony*). If $a \longrightarrow x$, then $(\exists x) : a \longrightarrow x$ (*zig*), so $a \sim a$.

2. *Symmetry*: Let's assume that $a \sim b$. Then, b and a satisfy the same atoms (by *atomic harmony*). In addition, if $a \longrightarrow x$ and $(\exists y) : b \longrightarrow y$, then: if $b \longrightarrow y$, then $(\exists x) : a \longrightarrow x$ (*zig*), so $b \sim a$.

3. *Transitivity*: Let's assume that $a \sim b$ si $b \sim c$. Then, $a \longrightarrow x$, $b \longrightarrow y$, $c \longrightarrow z$. Then, if $a \longrightarrow x$ and $c \longrightarrow z$, we have that $a \sim c$. $\square$

b) If $\sim$ is a bisimulation on $\mathfrak{M}$, then its inverse: $\sim^{-1}$ is also a bisimulation on $\mathfrak{M}$.

Proof. We have that: $a \sim b$ and $\sim^{-1} = \{(b,a) : a \sim b\}$. From $a \sim b$ we deduce that: (**1**) $\forall p \in \mathsf{Atoms}(\mathfrak{L}) : a \models p \Leftrightarrow b \models p$, (**2**) *zig*: $a \longrightarrow a' \Rightarrow \exists b' : b \longrightarrow b'$ and $a' \sim b'$, (**3**) *zag*: $b \longrightarrow b' \Rightarrow \exists a' : a \longrightarrow a'$ and $a' \sim b'$. We arbitrarily choose a and b, and proceed to show $b \sim^{-1} a$ is a bisimulation as follows:

Atomic harmony: From (**1**), we immediately obtain that $\forall p \in \mathsf{Atoms}(\mathfrak{L}) : b \models p \Leftrightarrow a \models p$.

Zig: *zig* for $b \sim^{-1} a$ is identical with proposition (**3**) (*zag* for $a \sim b$).

Zag: The same, condition *zag* for $b \sim^{-1} a$ is identical with proposition (**2**) (*zig* for $a \sim b$).

In another approach, $a \sim b \Rightarrow b \sim a$ is true because bisimulation is a symmetrical relation (being an equivalence relation), so $a \sim^{-1} b$ is a bisimulation. $\square$

c) If $\sim$ and $\equiv$ are bisimulation on $\mathfrak{M}$, then their composition: $\sim \circ \equiv$ is also a bisimulation on $\mathfrak{M}$.

Proof. If $\sim \subseteq X \times Y$, $\approx \subseteq Y \times Z$ (bisimulations), then $\approx \circ \sim \subseteq X \times Z$, $\approx \circ \sim = \{(x,z) : \exists y (x \sim y \wedge y \approx z)\}$. We will only prove the *atomic harmony* and *zig*.

Atomic harmony: Because it holds that $x \sim y$, we have that: $\forall p \in \mathsf{Atoms}(\mathfrak{L}) : x \models p \Leftrightarrow y \models p$. Also, because $y \approx z$, we have that $\forall p \in \mathsf{Atoms}(\mathfrak{L}) : y \models p \Leftrightarrow z \models p$, so: $\forall p \in \mathsf{Atoms}(\mathfrak{L}) : x \models p \Leftrightarrow z \models p$.

Zig: If $x \sim y$, $x \longrightarrow x' \Rightarrow \exists y : y \longrightarrow y'$ and $x' \sim y'$ and if $y \approx z$, $y \longrightarrow y' \Rightarrow \exists z' : z \longrightarrow z'$ and $y' \approx z'$. Because we have that $x \longrightarrow x'$ and $z \longrightarrow z'$ and $x' \sim y'$, $y' \approx z'$ (so $(x',z') \in \approx \circ \sim$), it results that the condition *zig* is verified by $\approx \circ \sim$. $\square$

d) If $\{\sim_i : i \in I\}$ is a set of bisimulations on $\mathfrak{M}$, then their union: $\bigcup_{i \in I} \sim_i$ is a bisimulation on $\mathfrak{M}$.

Proof: We'll show that for any $\sim_1$ and $\sim_2$ bisimulation on $\mathfrak{M}$, it holds that: $\sim_1 \cup \sim_2$ is a bisimulation on $\mathfrak{M}$.

First, let's remember that: $\sim_1 \cup \sim_2 = \{(x,y) \in \sim_1 \times \sim_2 \mid (x,y) \in \sim_1 \vee (x,y) \in \sim_2\}$.

We'll assume $w(\sim_1 \cup \sim_2)u$ and show the following:

1) *Atomic harmony:* From our assumption, it results that $w \sim_1 u$ or $w \sim_2 u$; Also from the assumption - the fact that $\sim_1$ and $\sim_2$ are bisimulations - we conclude that in both cases w and u satisfy the same atoms;

2) *Zig*: if $(w,u) \in \sim_1 \cup \sim_2$, then we distinguish two cases: **a**) if $w \sim_1 u$ and $w \longrightarrow w'$, then $\exists u' : u \longrightarrow u'$ and $w' \sim_1 u'$, by which: $w'(\sim_1 \cup \sim_2)u'$; and **b**) if $w \sim_2 u$ and $w \longrightarrow w'$, then $\exists u' : u \longrightarrow u'$ and $w' \sim_2 u'$, and we conclude that: $w'(\sim_1 \cup \sim_2)u'$.

3. *Zag*: We reason by analogy with *zig*. $\square$

e) The empty relation is a bisimulation.
Proof: It is evident. By the fact that the empty bisimulation doesn't contain any relation between states, the conditions of bisimulation are satisfied by default.

Given the property expressed by **(d)**, we can define *the largest bisimulation* on a model. Let's consider $\approx$, a bisimulation that relates or links all states of W. Then, $B = \bigcup_{\approx \subseteq W \times W} \approx$ is the union of all bisimulations and a bisimulation itself. B is the largest bisimulation on $\mathfrak{M}$.

We can say that bisimulations (...and the equivalent notion of a *p-morphism*) are weaker than isomorphisms. In what sense? Isomorphic models, even though satisfying the same propositions, do not allow for any change of structure (different domains of possible worlds and accessibility relations). As an advantage, the bisimulation relation may hold between models with very different domains and relations but satisfying the same formulas.

It is worthy of another commentary that, dependending on the field of study, bisimulation has different, although related, uses and theoretical meanings. In automata theory, a bisimulation is an equivalence relation between distinct (not isomorphic) *transition systems* of similar behaviour. In set theory without the *axiom of foundation* - which tells us that a set cannot contain itself as a subset - the identity or equality of two sets cannot be defined extensionally, like this: two sets are equal if they contain the same elements. The reason is simple: if a set contains itself as a subset, then its depth is infinite, so induction could not be used to establish its equality to other sets (see [14], p. 133). Actually, the equality of such unusual sets is proved by showing that they are bisimilar (there is a bisimulation between them).

3 The Minimal Model

Bibliographical indications: Wang, Y., *Epistemic Modelling and Protocol Dynamics*, p. 12., Van Benthem, J., *Modal Logic for Open Minds*, pp. 27-28.

Let $\mathfrak{M}$ be a model that satisfies a set φ of propositions. We can ask ourselves: what is the smallest model that preserves the same information? In our understanding, a minimal model (the one used and analysed in this section will be a model factorized by the largest auto-bisimulation. This construction is also entitled the *bisimulation contraction* or the *maximally contracted model* or a *strongly extensional model*. In automata theory we find an analogous structure named *quotient transition system*, and also in graph theory the structure named *quotient graph*.

We'll use the following notation for our minimal model: $\mathfrak{M}/_{\sim} = (W/_{\sim}, \longrightarrow_{\sim}, \pi_{\sim})$. Its construction needs a few other preliminary constructions and some steps to follow (each step corresponding to a stage of its construction).

a) $\mathfrak{M}/_{\sim}$ is the quotient model - the model factorized by the relation $\sim$. Remember that $\sim$ is the largest bisimulation on $\mathfrak{M}$. Once we've found a relation $\sim \subseteq W \times W$, we can proceed:

b) $W/_{\sim}$, is the set of all equivalence (modulo $\sim$) classes of our states of W. So:

(3.1) $W/_{\sim} = \{[w]_{\sim} \mid w \in W\}$, where:

(3.1.1) $[w]_{\sim} = \{w' \in W \mid w \sim w'\}$

c) To obtain $\longrightarrow_{\sim} \subseteq W/_{\sim} \times W/_{\sim}$, we use the following rule:

(3.2) $w \xrightarrow{a} w' \Longrightarrow [w]_{\sim} \xrightarrow{a}_{\sim} [w']_{\sim}$

A little more informal, if two states w and w' are linked by the relation $\xrightarrow{a}$, then their quotients (modulo $\sim$) ($[w]_{\sim}$ and $[w']_{\sim}$) are linked by the relation $\xrightarrow{a}_{\sim}$.

d) The valuation function $\pi_{\sim}$ is defined by taking into account that $W_{\sim}$ is a set of equivalence classes of elements of W. So:

(3.3) $\pi_{\sim} : \mathsf{Atoms}(\mathfrak{L}) \longrightarrow 2^{W/_{\sim}}$

So we have the following definition of the valuation function $\pi_{\sim}$:

(3.4) $\pi_{\sim}(p) = \{[\pi(p)]_{\sim}\}$

Because $\mathfrak{M}/_{\sim}$ is obtained by factoring the model $\mathfrak{M}$ by $\sim$, the two models are bisimilar, and therefore equivalent (in the sense that they contain the same information).

Example:

Let's consider the Kripke-model $\mathfrak{M} = (W, \longrightarrow_a, \pi)$, defined in the following manner: $W = \{1,2,3,4\}$, $\longrightarrow_a = \{(1,2),(1,3),(2,4),(3,4)\}$. The largest bisimulation on the model $\mathfrak{M}$ is constructed out of the following bisimulations: $1 \sim 1$, $2 \sim 3$, $4 \sim 4$. As we defined it above, the largest bisimulation is the union of bisimulations: $\sim = \{(1,1),(2,3),(4,4)\}$. Because bisimulation is an equivalence relation (see the proof above), $\mathfrak{M}$ can be factored by $\sim$: $\mathfrak{M}/_\sim = (W_\sim, \longrightarrow_\sim, \pi_\sim)$ is defined as below:

The domain. $W/_\sim = \{[1]_\sim, [2]_\sim, [4]_\sim\}$. Note that $[2]_\sim = [3]_\sim = \{2,3\}$, because $2 \sim 3$. In a sense, 2 and 3 of W are "contracted" into a single world in $W_\sim$. No such a contraction is possible with 1 and 4 because they're only bisimilar with themselves.

The accessibility relation. Let's see how the quotient accessibility $\longrightarrow_\sim$ relation is constructed using the rule presented above:

$1 \longrightarrow 2 \Longrightarrow [1]_\sim \longrightarrow_\sim [2]_\sim$

$1 \longrightarrow 3 \Longrightarrow [1]_\sim \longrightarrow_\sim [2]_\sim$

$2 \longrightarrow 4 \Longrightarrow [2]_\sim \longrightarrow_\sim [4]_\sim$

$3 \longrightarrow 4 \Longrightarrow [2]_\sim \longrightarrow_\sim [4]_\sim$

World 1 is linked with 3, so $[1]_\sim$ is linked with $[3]_\sim$, but because $[3]_\sim = [2]_\sim$, this is equivalent to linking $[1]_\sim$ with $[2]_\sim$. The same with $[3]_\sim$ and $[4]_\sim$.

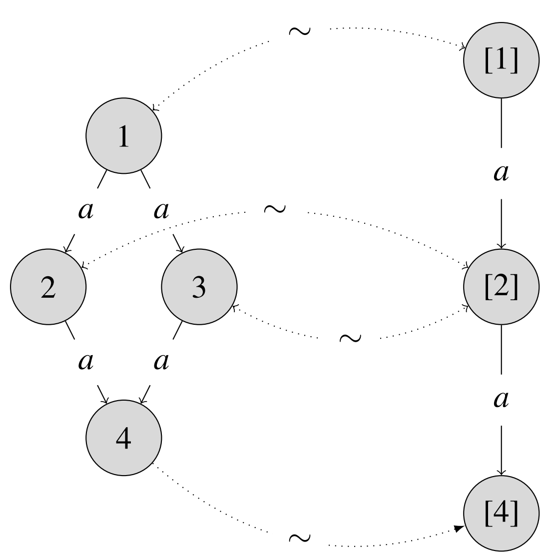

Figure 2 presents, on the left, the models $\mathfrak{M}$, and on the right the quotient model $\mathfrak{M}/_\sim$. The pointed arrows between the models represent the bisimulations $\sim$ between the states of the two

models.

Results

Let $\mathfrak{M} = (W, \longrightarrow, \pi)$ be a models and $\sim$ its largest bisimulation. Then, $\mathfrak{M}/_\sim = (W/_\sim, \longrightarrow_\sim, \pi_\sim)$ is the quotient of the model by $\sim$. The relation $\Delta = \{(x,x) \mid x \in W\} = \{(x,y) \mid x = y, x \in W, y \in W\}$ (the *diagonal* of $W \times W$), a bisimulation on $\mathfrak{M}$. From the way it is defined, Δ *bisimilates* (links by a bisimulation) every state to itself! As a consequence, I will refer to it by the term *the diagonal bisimulation* or *the diagonal of the model*.

Proposition. The diagonal (Δ) of our model $\mathfrak{M}$ is a bisimulation (*the diagonal bisimulation*) on $\mathfrak{M}$.

Proof. Let's consider $\Delta = \{(x,x) \mid x \in W\}$. We have to show that Δ verifies the three conditions for being a bisimulation.

1. *Atomic harmony*: Let us assume that $(x,x) \in \Delta$. Then, $x \models p$ iff $x \models p$ is evidently true.

2. *Zig*. Let us assume that $(x,x) \in \Delta$ and $x \longrightarrow y$. Then, $(\exists y') : x \longrightarrow y'$ for $y' = y$ and $(y,y) \in \Delta$ from the definition of our relation Δ.

3. *Zag*. The argument is symmetrical to *Zig*. $\square$

Proposition. Every bisimulation on the minimal model (the quotient model) is included in the *diagonal bisimulation*. Formally:

Let us consider the model $\mathfrak{M}$ and $\sim$ the largest bisimulation on it and its quotient model $\mathfrak{M}/_\sim$. Also, $\approx \subseteq W/_\sim \times W/_\sim$ a bisimulation on $\mathfrak{M}/_\sim$ and $\Delta \subseteq W/_\sim \times W/_\sim$ the diagonal bisimulation. Then: $\approx \subseteq \Delta$, for $\forall \approx \subseteq W/_\sim \times W/_\sim$.

What does this mean? That any possible bisimulation on the minimal model links states to themselves. To observe the difference, the bisimulation on the non-minimal model of **Figure 2** (see above) links the distinct states **2** and **3**.

Proof(See [12]). We'll consider w and u two states of $\mathfrak{M}$, $[w]_\sim$ and $[u]_\sim$ their equivalence classes *modulo* $\sim$ (so $[w]_\sim \in \mathfrak{M}/_\sim$ and $[u]_\sim \in \mathfrak{M}/_\sim$). Let $\approx$ be a bisimulation between $[w]_\sim$ and $[u]_\sim$. In order to show that $\approx$ is included in the diagonal, we have to show that $[w]_\sim = [u]_\sim$, and this can be proved easily by showing that $w \sim u$ (remember a property of equivalence classes: $[a]_\sim = [b]_\sim \Leftrightarrow a \sim b$).

Let's consider a relation $\equiv$ on $\mathfrak{M}$ ($\equiv \subseteq W \times W$) between any w and u defined as such:

$w \equiv u \Leftrightarrow [w]_\sim \approx [u]_\sim$

Let's prove that $\equiv$ is a bisimulation. We'll suppose (to prove *zig*[1]) that $w \longrightarrow w'$. Because $\approx$ is a bisimulation, we have that if $[w]_\sim \longrightarrow_\sim [w']_\sim$ then it exists $[x]_\sim$ so that: $[u]_\sim \longrightarrow_\sim [x]_\sim$. By the definition of equivalence classes, we have that it exists $u' : u' \sim u$. Because $[u]_\sim \longrightarrow_\sim [x]_\sim$, we have that it exists a state x' so that $u' \longrightarrow x'$. Because $\sim$ is a bisimulation, we obtain (*zag*) that it exists a state x'' so that $u \longrightarrow x''$ and $x'' \sim x'$. Since it holds that $u \longrightarrow x''$, we proved that $w \equiv u$ is a bisimulation. Now, because $\equiv \subseteq \sim$ ($\equiv$ is included in the largest bisimulation on the models, $\sim$), we have that $w \equiv u \Longrightarrow w \sim u$, what was to be demonstrated. $\square$

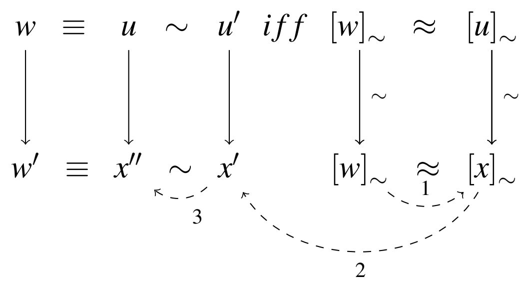

Figure 3: To ease the understanding of the proof, I attached a graphic representation of the accessibility relations ($\longrightarrow$ and $\longrightarrow_\sim$) and bisimulations ($\equiv$, $\sim$, $\approx$) between states and classes.

4 Characteristic Formulas

We'll begin by defining a characteristic formula of a possible world following [7] and [2] and afterwards we'll discuss its meaning.

$$\delta^0_{\mathfrak{M},w} := \bigwedge\{p \mid \mathfrak{M}, w \models p\} \wedge \bigwedge\{\neg p \mid \mathfrak{M}, w \not\models p\}$$

$$\delta^{n+1}_{\mathfrak{M},w} := \delta^0_{\mathfrak{M},w} \wedge \bigwedge_{wRu} \Diamond \delta^n_{\mathfrak{M},u} \wedge \Box \bigvee_{wRu} \delta^n_{\mathfrak{M},u}$$

As can be seen, a characteristic formula of w is inductively constructed out of all the formulas satisfied at w, from atomary ones to more complex, modalized ones. The need for an inductive definition is given by the fact that if a world w sees a world u, then w must consider as possible all formulas satisfied at u, and if u sees v, then w considers all formulas of v as ”possibly possible”, and so on...

[1]The argument for *zag* is symmetrical.

An important result (see [7] and [2] for a proof) tells that a possible world w satisfies the characteristic formula of another possible world if and only if the two are bisimilar:

Proposition. The following are equivalent:

(1) $\mathfrak{M}, w \models \delta_u^n$

(2) $(\mathfrak{M}, w) \sim (\mathfrak{M}, u)$

Proof: See [7] and [2].

5 Public Announcement Logic

What is particular about Public Announcement Logic (hereafter: PAL), discovered independently by J. Plaza [13] and J. Gerbrandy [6], is that its semantics is based on model-transformers, functions that change a Kripke-model into another one, of a different domain, accessibility relation and valuation. Its use was to describe the evolution of knowledge of a group of agents after receiving certain, truthful information. These matters are presented and discussed in length in [17]. PAL introduces a binary operator $\langle \cdot \rangle \cdot$ that relates the two models and states what is acquired after the announcement: $\langle \phi \rangle \psi$ is read: *after a public announcement of formula* ϕ*,* ψ *is true*. The semantics is the following:

$\mathfrak{M}, w \models \langle \phi \rangle \psi$ iff $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}|_\phi, w \models \psi$, where $\mathfrak{M}_\phi = (W|_\phi, \rightarrow|_\phi, \pi|_\phi)$ is defined as following:

(1) $W|_\phi = \{w \mid \mathfrak{M}, w \models \phi\}$

(2) $\rightarrow|_\phi = \rightarrow \cap (W|_\phi \times W|_\phi)$

(3) $\pi|_\phi = \pi \cap W|_\phi$

As can be seen, the new domain, $W|_\phi$, is composed only of states satisfying ϕ, and the accessibility relation and valuation are restricted to $W|_\phi$.

The matters of completeness are elegantly solved by reduction axioms that translate every PAL-formula into an S5-formula. For complete proofs, see [9].

6 Talking about truths at possible worlds

For starters, let's see what happens after announcing a characteristic formula. This is a trick used in [19] in order to restrict the domain to a singleton and prove an interesting result on dynamic epistemic logics with public assignments (or substitution). Remember that a chacteristic formula of w is only true at w and its bisimilars, since bisimilar states satisfy the same formulas. Now, let's assume that we work only on maximally contracted models: models in which all worlds are bisimilar only with themselves. In this case, since a public announcement of, say, ϕ restricts the domain to states in which ϕ is satisfied, after announcing δ_w, the characteristic formula of w, the model will be restricted to a singleton domain containing only w.

The following formula allows us to "test" whether ϕ is true at world u: $\langle\delta_u\rangle\phi$. But we can use it to talk at w about the truth of ϕ at world u: $\mathfrak{M}, w \models \Diamond\langle\delta_u\rangle\phi$. Let's see why: $\Diamond\langle\delta_u\rangle\phi$ is true iff there is a world x such that $w \longrightarrow x$ and $\mathfrak{M}, x \models \langle\delta_u\rangle\phi$. This happens iff $\mathfrak{M}, x \models \delta_u$ and $\mathfrak{M}|_{\delta_u}, x \models \phi$. Now, note that the only x that satisfies δ_u is u and the model is restricted to this world only. Because we assumed that we work on maximally-contracted models, there is no other world satisfying it, so ϕ is evaluated at u, the intended world for its evaluation. So we have this equivalence:

$\mathfrak{M}, w \models \Diamond\langle\delta_u\rangle\phi$ iff $\mathfrak{M}, u \models \phi$

Unfortunately, the above equivalence does not hold if ϕ is a modalized formula: imagine that at a world w it holds that $\neg\phi$ and $\Diamond\phi$ because wRu and at world u it is true that ϕ; then, after announcing the characteristic formula of w, world u will be eliminated, therefore $\Diamond\phi$ will no longer be true at w.

Miroiu (see [11]) sets up a modal language in which one can refer to possible worlds. For example, the formula $wu\phi$ is accepted in the language and is read "at w it is true that ϕ is satisfied at u". Further, he listed a series of intuitive principles governing his logical system: for example, if at w it is false that ϕ, then it is false that at w it is true that ϕ, or: if at w the formula $\phi \wedge \psi$ is satisfied, then it is true that ϕ is true at w and ψ is true at w. These principles can be shown as valid in PAL using carefully chosen annoucements of characteristic formulas. However, formulas like $w\Diamond\phi$, read "at w it is possible that ϕ", will raise the following difficulty: after announcing δ_w, the model will be restricted to w only and will lose the link to the ϕ-satisfying world(s) (assuming that ϕ is not true at w, because reflexivity of the accesibility relation implies the possibility of *de facto* truths). Another problem comes from trying to obtain formulas that say "at w it is true that at u it is true that ϕ" (or "at w it is true that ϕ" is true at u), easily representable in Miroiu's language as: $wu\phi$. Intuitively, this would be done by $\Diamond\langle\delta_w\rangle\Diamond\langle\delta_u\rangle\phi$, but after announcing δ_w we'll remain with a singleton domain composed of w only and lose the possibility to evaluate ϕ at u. But the way out of this trouble is easy: the first announcement should be a disjunction of δ_w and δ_u, and the second announcement should be the one to restrict the domain to u and evaluate ϕ there: $\Diamond\langle\delta_w \vee \delta_u\rangle\Diamond\langle\delta_u\rangle\phi$.

7 Expressing maximal contraction in the language

In this section we'll see how we can express a model's property of being maximally contracted (or minimal modulo the largest bisimulation) as a formula in the language of PAL. One interesting consequence is that checking whether a certain model is maximally contracted is reduced to verifying the truth of a certain formula at a world in our model.

Using the trick of announcing characteristic formulas, we can find a formula that states the property of maximally-contraction as a formula of the PAL-language. If this formula is true, then the model is minimal (of course, in the sense of being maximally-contracted). Why is that? Because the condition of minimality boils down to this observation: for any two worlds, none of them satisfies the characteristic formula of the other; if that would be the case, then the two would be bisimilar. Formally: $\mathfrak{M}, w \models \bigwedge_{x \neq y} \neg\Diamond\langle\delta_x\rangle\delta_y$.

8 Final remarks

Observe that this alternative way of referring to truths at possible worlds differs from Hybrid logic and Miroiu's logical system in that the latters use ”nominals”, propositional atoms that refer directly to possible worlds. We've been using characteristic formulas in order to refer to possible worlds, but their use does not solicit the introduction of ”special” atoms in the ordinary modal language. In a sense, the difference is analogous to that between directly referring proper names and descriptions; characteristic formulas act like descriptions in the sense that they describe a possible world, but the same description might pick out two worlds. We tackled this problem by working with only minimal or maximally contracted models. But this is just a sketch, since I haven't offered an algorithm or a translation schema from sentences of these languages to the language of PAL and solutions to the problems raised at the end of the 6th section.

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